THE SOLUTION OF THE PROBLEM OF THE SIMPLE OSCILLATOR BY A COMBINATION OF THE SCHROEDINGER AND THE LANCZOS THEORIES

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Synopsis.—Schroedinger has recently published a new quantum condition in form of a variation principle. This leads to a differential equation for a certain function analogous to the Hamilton-Jacobi, S, which possesses relatively simple solutions for certain discrete values of the energy constant. Schroedinger has shown that in the case of hydrogen, these discrete values are those of the Bohr levels, provided that the single arbitrary constant of his theory (which must be introduced because of dimensional considerations) is given the value $h/2\pi$.

The purpose of this paper is to show, in a purely formal way, that the Schroedinger equation *must be the basis for the Born, Jordan and Heisenberg matrix calculus*. This is most readily accomplished by using the Lanczos modification of the latter.

Lack of space prevents any general discussions, hence, only a very formal solution of the special problem of the simple linear oscillator is attempted. All results are capable of immediate generalization, however. The general theory, and an attempt to interpret some of the postulates, will be published in another place.

The Solution of the Problem.—The Lanczos[1] quantum theory requires a set of orthogonal functions, by means of which the matrices of Born and Jordan[2] are to be determined. Thus far, no problems have been solved by means of this theory, primarily because the determination of the orthogonal functions has been impossible.

Schroedinger[3] has published a quantum principle which leads directly to a set of orthogonal functions. The purpose of this paper is to show that if these are interpreted as the functions entering into the Lanczos theory, the matrices of the Born-Jordan theory are readily obtained.

We limit ourselves, for brevity, to the problem of the simple oscillator. The Hamilton-Jacobi equation for this problem is

$$G = \frac{1}{2\mu}\left(\frac{dS}{dx}\right)^2 + \frac{1}{2}\mu\omega^2x^2 - W = 0. \qquad (1)$$

Following Schroedinger, we substitute $S = h/2\pi.\log\psi$ and inquire, not for a solution of equation 1, but for a solution of

$$\delta\int_{-\infty}^{+\infty}\psi^2 G dx = 0 \qquad (2)$$

W being constant during the variation, and $\delta\psi$ vanishing at $x = \pm\infty$. It may be shown that the solution of this problem is equivalent to the solution of

$$-\frac{1}{2\mu}\frac{h^2}{4\pi^2}\frac{d^2\psi}{dx^2} + \left\{\frac{1}{2}\mu\omega^2x^2 - W\right\}\psi = 0. \tag{3}$$

The solution of this equation has been discussed by P. S. Epstein[4] and leads to parabolic cylinder functions. Let $x = \sqrt{\frac{h}{4\pi\mu\omega}}\,u$; then (3) becomes

$$\frac{d^2\psi}{du^2} - \left\{\frac{u^2}{4} - \frac{2\pi W}{h\omega}\right\}\psi = 0 \tag{3a}$$

which possesses solutions of the forms

$$e^{-u^2/4}f(u) \text{ and } e^{+u^2/4}F(u) \tag{4}$$

where

$$\begin{aligned}\frac{d^2f}{du^2} - u\frac{df}{du} + \left\{\frac{2\pi W}{h\omega} - \frac{1}{2}\right\}f &= 0\\ \frac{d^2F}{du^2} + u\frac{dF}{du} - \left\{\frac{2\pi W}{h\omega} - \frac{1}{2}\right\}F &= 0.\end{aligned} \tag{5}$$

The solution of these equations leads, in general, to infinite power series in u, but provided that

$$\frac{2\pi W}{h\omega} - \frac{1}{2} = \text{positive integer}$$

or

$$W_n = \left(n + \frac{1}{2}\right)\frac{h\omega}{2\pi} \tag{6}$$

the solutions are finite polynomials.

We now seek for those solutions of (3) which satisfy the conditions of orthogonality

$$\begin{aligned}\int_{-\infty}^{+\infty}\psi_n(u)\,\psi_m(u)dx &= 0 \quad \text{if } n \neq m\\ &= 1 \quad \text{if } n = m\end{aligned} \tag{7}$$

and find that

$$\psi_n = \sqrt{n!}\,\sqrt[4]{\frac{2\mu\omega}{h}}\,e^{-u^2/4}f_n(u) \tag{8}$$

provided f_n is the polynomial determined by

$$\begin{aligned}f_{n-2} - uf_{n-1} + nf_n &= 0\\ f_0 = 1 \qquad f_1 &= u.\end{aligned} \tag{9}$$

From (8) and (9), we may deduce the following very important formula:

$$x\psi_n(x) = \sqrt{\frac{h}{4\pi\mu\omega}}\left\{\sqrt{n}\,\psi_{n-1}(x) + \sqrt{n+1}\,\psi_{n+1}(x)\right\}. \quad (10)$$

The "equations of motion" of the simple oscillator are

$$\begin{aligned} DX - XD &= \frac{1}{\mu} P \\ DP - PD &= -\mu\omega^2 X \end{aligned} \quad (11)$$

where P and X are operators satisfying the "quantum condition"

$$PX - XP = h/2\pi i \quad (12)$$

and the mode of operation of any operator is defined by an equation of the type

$$Pf(x) = \int_{-\infty}^{+\infty} P(x\xi)\, f(\xi) d\xi. \quad (13)$$

The problem is to determine the functions $P(x\xi)$, etc.

The essential mathematical point of this paper is to show that the condition (12) may be replaced by the requirements

$$\begin{aligned} x &\equiv \frac{1}{\psi_n(x)} \int_{-\infty}^{+\infty} X(x\xi)\, \psi_n(\xi) d\xi \\ \frac{2\pi i W_n}{h} &= \frac{1}{\psi_n(x)} \int_{-\infty}^{+\infty} D(x\xi)\, \psi_n(\xi) d\xi. \end{aligned} \quad (14)$$

These equations have an important interpretation which cannot be explained without an involved discussion of the operator calculus, which must be postponed until another paper.

If we expand $D(x\xi)$, $X(x\xi)$, etc., in terms of $\psi_j(x)$ and $\psi_k(\xi)$, which is, in general, possible because of (7), we should obtain double infinite series of form:

$$\begin{aligned} D(x_1\xi) &= \sum_{jk} D_{jk}\, \psi_j(x)\, \psi_k(\xi) \\ X(x\xi) &= \sum_{jk} X_{jk}\, \psi_j(x)\, \psi_k(\xi). \end{aligned} \quad (15)$$

The coefficients of these series will be shown to be the matrices of the Born-Jordan calculus. The last of (14) now becomes

$$\begin{aligned} \frac{2\pi i W_n}{h} &= \frac{1}{\psi_n(x)} \sum_{jk} D_{jk}\, \psi_j(x) \int_{-\infty}^{+\infty} \psi_k(\xi)\, \psi_n(\xi) d\xi \\ &= \frac{1}{\psi_n(x)} \sum_{j} D_{jn}\, \psi_j(x) \qquad \text{by (7).} \end{aligned}$$

Hence

$$D_{jk} = 0 \qquad j \neq k$$
$$D_{jj} = \frac{2\pi i W_j}{h}. \tag{16}$$

Similarly, the first of (14) becomes

$$x \equiv \frac{1}{\psi_n(x)} \sum_j X_{jn}\, \psi_j(x)$$

or

$$x\psi_n(x) \equiv \sum_j X_{jn}\, \psi_j(x). \tag{17}$$

Comparing this with (10), we see that

$$X_{jn} = 0, \qquad j \neq n \pm 1$$
$$X_{n+1,\,n} = \sqrt{\frac{(n+1)h}{4\pi\mu\omega}} \tag{18}$$
$$X_{n-1,\,n} = \sqrt{\frac{nh}{4\pi\mu\omega}}.$$

If we now carry out the operations indicated in the first of (11), we find that

$$P(x\xi) = \sum_{nm} P_{nm}\, \psi_n(x)\, \psi_m(\xi)$$

where

$$P_{nm} = 0, \qquad n \neq m \pm 1$$
$$P_{n+1,\,n} = i\omega X_{n+1,\,n} \tag{19}$$
$$P_{n-1,\,n} = i\omega X_{n-1,\,n}.$$

Since these matrices are identical with those obtained as the solution of (11) and (12), using the matrix calculus, it is obvious that the conditions (12) are satisfied, as may also be verified by substitution. This was the mathematical result which we set out to obtain. The generalization of this method of solving problems in the Born-Jordan matrix calculus, and a discussion of its significance will shortly be published elsewhere. An attempt to solve the problem of the hydrogen atom completely is also under way. It may be remarked that the solution of Schroedinger's equation at once brings us to the stage reached by Pauli by the elaborate computations of the matrix calculus.

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[1] Kornel Lanczos, *Zs. Physik*, **35**, 812 (1926).

[2] Born and Jordan, *Ibid.*, **34**, 858 (1925); and numerous other papers in the same journal.

[3] E. Schroedinger, *Ann. Physik*, **79**, 361 (1926).

[4] P. S. Epstein, "Beugung an einem ebenen Schirm," Dissertation, Munich, 1914. I am greatly indebted to Professor Epstein, not only for this reference, but for much other assistance and encouragement in the course of this investigation.